

Section 9.4 - Vector and Scalar functions and their fields

We define a vector function \underline{v} , whose values are vectors,

$$\underline{v} = \underline{v}(P) = [v_1, v_2, v_3]$$

We can also define a scalar function f , whose values are scalars, $f = f(P)$

Example 1: Euclidean metric.

$$f(P) = f(x, y, z) = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$$

Example 2: Vector field.

$$\begin{aligned}\underline{v}(P) &= \underline{v}(x, y, z) = \underline{\omega} \times \underline{r} = \omega \hat{k} \times (x\hat{i} + y\hat{j} + z\hat{k}) \\ &= \omega[-y, x, 0] = -\omega y\hat{i} + \omega x\hat{j}\end{aligned}$$

Derivative of a vector function

A vector function $\underline{v}(t)$ is said to be differentiable at a point t if the limit exists

$$\underline{v}'(t) = \lim_{\Delta t \rightarrow 0} \frac{\underline{v}(t + \Delta t) - \underline{v}(t)}{\Delta t}$$

In components, $\underline{v}'(t) = [v_1'(t), v_2'(t), v_3'(t)]$

Some properties:

$$(c\underline{v})' = c\underline{v}'$$

$$(\underline{u}, \underline{v})' = (\underline{u}', \underline{v}) + (\underline{u}, \underline{v}')$$

Example: Let $\|\underline{v}(t)\| = c$, but

$$\|\underline{v}(t)\|^2 = (\underline{v}, \underline{v}) = c^2$$

$$\frac{d}{dt} \|\underline{v}(t)\|^2 = 0 = (\underline{v}', \underline{v}) + (\underline{v}, \underline{v}') = 2(\underline{v}, \underline{v}')$$

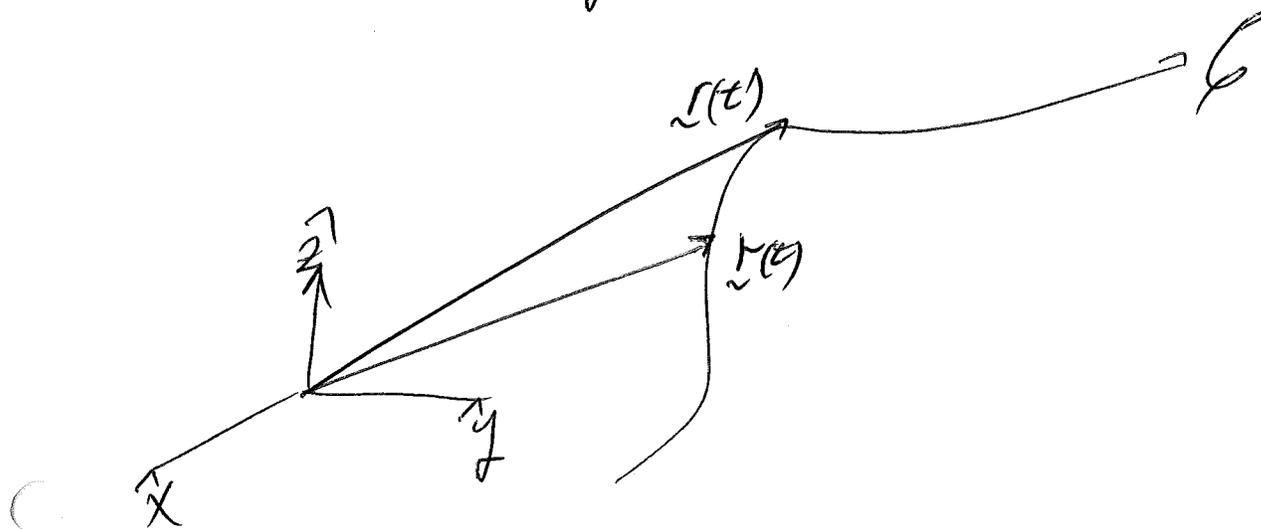
$$\Rightarrow \underline{v}, \underline{v}' \perp$$

Curves, arc length, curvature

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Parametric representation of C with t ,

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$$



Circle: let $x^2 + y^2 = 4$, center $(0,0)$, radius $r=2$;

$$\vec{r}(t) = [2 \cos t, 2 \sin t, 0] \quad , \quad 0 \leq t \leq 2\pi$$

$$\text{So } x^2 + y^2 = 4 \cos^2 t + 4 \sin^2 t = 4 \quad \checkmark$$

Ellipse: The function $\vec{r}(t) = a \cos t \vec{i} + b \sin t \vec{j}$

$$\text{Since } \sin^2 t + \cos^2 t = 1, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Tangent to a curve.

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Let C be given by $\underline{r}(t)$ and P, Q correspond to t and $t + \Delta t$, then a vector in the direction of C is

$$\frac{1}{\Delta t} [\underline{r}(t + \Delta t) - \underline{r}(t)], \text{ take } \lim \Delta t \rightarrow 0$$

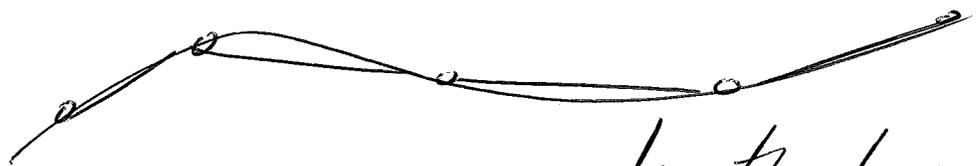
$$\underline{r}'(t) = \lim_{\Delta t \rightarrow 0} \left[\frac{\underline{r}(t + \Delta t) - \underline{r}(t)}{\Delta t} \right], \text{ if } \underline{r}'(t) \neq 0,$$

Then $\underline{r}'(t)$ is called the tangent vector of C at P . Unit tangent vector is defined

$$\text{or } \underline{u} = \frac{\underline{r}'}{\|\underline{r}'\|}$$

Length of a curve:

$$L = \int_a^b \sqrt{(\underline{r}', \underline{r}')} dt$$



length of a curve

Differentiating,

$$\left(\frac{ds}{dt}\right)^2 = (\underline{r}', \underline{r}') = \|\underline{r}'(t)\|^2$$

$$\text{If } d\underline{r} = dx \underline{i} + dy \underline{j} + dz \underline{k}$$

$$ds^2 = (d\underline{r}, d\underline{r}) = dx^2 + dy^2 + dz^2,$$

ds is the linear element of C .

Scalar, vector, tensor functions.

A tensor function is a function whose arguments are one or more tensor variables and whose values are scalar, vectors, or tensors.

$$\exists f: \underline{A} \rightarrow \begin{array}{l} \alpha \in \mathbb{R} \\ \underline{u} \in \mathbb{E} \\ \underline{A} \in \mathcal{H} \end{array}$$

$\underline{\Phi}(\underline{B})$ scalar-valued

$\underline{u}(\underline{B})$ vector-valued

$\underline{A}(\underline{B})$ tensor-valued

Consider scalar functions of one scalar variable.

$$\Phi = \Phi(t), \quad \underline{u} = \underline{u}(t) = u_i(t) \underline{e}_i,$$

$$\underline{A} = \underline{A}(t) = A_{ij}(t) \underline{e}_i \otimes \underline{e}_j \quad ; \quad \{ \underline{e}_i \} \text{ are fixed!}$$

Therefore, the first derivative of \underline{u} and \underline{A} wrt time

$$\dot{\underline{u}} = \dot{u}_i(t) \underline{e}_i \quad ; \quad \dot{\underline{A}} = \dot{A}_{ij}(t) \underline{e}_i \otimes \underline{e}_j$$

Identities

$$\dot{\underline{u}} \pm \dot{\underline{v}} = \dot{\underline{u}} \pm \dot{\underline{v}} ;$$

$$\dot{\underline{\Phi}} \underline{u} = \dot{\underline{\Phi}} \underline{u} + \underline{\Phi} \dot{\underline{u}} ;$$

$$\dot{\underline{u}} \otimes \underline{v} = \dot{\underline{u}} \otimes \underline{v} + \underline{u} \otimes \dot{\underline{v}} ;$$

$$\dot{\underline{A}} \pm \dot{\underline{B}} = \dot{\underline{A}} \pm \dot{\underline{B}} + \underline{A} \pm \underline{B}$$

$$\dot{\underline{A}}^T = \dot{\underline{A}}^T$$

$$\dot{\text{tr}} \underline{A} = \text{tr} \dot{\underline{A}}$$

gradient of a scalar-valued (tensor) function

Consider $\underline{\Phi}(\underline{A}) ; \underline{\Phi} : \underline{A} \rightarrow \mathbb{R}$

$\underline{\Phi}$ is nonlinear.

Taylor expand; $\underline{\Phi}(\underline{A} + d\underline{A}) = \underline{\Phi}(\underline{A}) + d\underline{\Phi} + \mathcal{O}(d\underline{A})$

$$d\underline{\Phi} =: \frac{\partial \underline{\Phi}(\underline{A})}{\partial \underline{A}} : d\underline{A} =: \text{tr} \left[\left(\frac{\partial \underline{\Phi}(\underline{A})}{\partial \underline{A}} \right)^T d\underline{A} \right]$$

Gradient of a tensor valued (tensor) function 3.-

Consider a nonlinear and smooth tensor-valued tensor function $\underline{A}(\underline{B}) \Rightarrow$

$$\underline{A} : \underline{B} \longrightarrow \underline{C}, \underline{C} \in \text{Sym.}$$

Compute the first order Taylor expansion

$$\underline{A}(\underline{B} + d\underline{B}) = \underline{A}(\underline{B}) + d\underline{A} + \mathcal{O}(d\underline{B})$$

$$d\underline{A} = : \frac{\partial \underline{A}(\underline{B})}{\partial \underline{B}} : d\underline{B} \quad ; \quad \text{the term}$$

$\frac{\partial \underline{A}(\underline{B})}{\partial \underline{B}}$ denotes the gradient

Gradients and related operators -

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Gradient of a scalar field:

$$\Phi(\underline{x} + d\underline{x}) = \Phi(\underline{x}) + d\Phi + \mathcal{O}(d\underline{x}^2)$$

$$d\Phi =: \frac{\partial \Phi}{\partial \underline{x}} \cdot d\underline{x} \quad \text{The total differential}$$

$$d\Phi = \frac{\partial \Phi}{\partial x_1} dx_1 + \frac{\partial \Phi}{\partial x_2} dx_2 + \frac{\partial \Phi}{\partial x_3} dx_3 \quad \text{or}$$

$$d\Phi = (\nabla \Phi, d\underline{x}) ;$$

$$\nabla \Phi = \text{grad } \Phi = \frac{\partial \Phi}{\partial x_i} \underline{e}_i$$

$$\text{i.e.} \quad \nabla(\cdot) = \frac{\partial (\cdot)}{\partial x_i} \underline{e}_i$$

Divergence of a vector field - The operator ∇ dotted with vector field $\underline{u}(\underline{x})$ is called the divergence of $\underline{u}(\underline{x})$

$$\text{div } \underline{u} =: (\nabla, \underline{u})$$

$$= \left(\frac{\partial(\cdot)l_1}{\partial x_1} + \frac{\partial(\cdot)l_2}{\partial x_2} + \frac{\partial(\cdot)l_3}{\partial x_3}, u_1 \underline{l}_1 + u_2 \underline{l}_2 + u_3 \underline{l}_3 \right)$$

$$= \left(\frac{\partial u_i}{\partial x_i} \underline{l}_j, \underline{l}_i \right) = \frac{\partial u_i}{\partial x_i}$$

$$= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

Curl of a vector field.

Gradient of a vector field. The gradient or derivative of a vector field is a second order tensor.

$$\text{grad } \underline{u} = \nabla \otimes \underline{u} = \left(\frac{\partial(\cdot)}{\partial x_j} \otimes \underline{l}_j \right) \underline{u}$$

$$= \frac{\partial u_i}{\partial x_j} \underline{l}_i \otimes \underline{l}_j$$

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Show: $\text{tr}(\text{grad } \underline{u}) = \text{div } \underline{u}$

The transposed gradient is given

$$\begin{aligned} \text{grad}^T \underline{u} &= \underline{u} \otimes \nabla = \underline{e}_i \otimes \frac{\partial \underline{u}}{\partial x_i} \\ &= \frac{\partial u_j}{\partial x_i} \underline{e}_i \otimes \underline{e}_j \end{aligned}$$

Divergence and gradient of a tensor.

A second order tensor dotted with the operator ∇ , denoted $\text{div } \underline{A}$ or $\nabla \cdot \underline{A}$ denotes the divergence of \underline{A} .

$$\begin{aligned} \text{div } \underline{A} &= (\nabla, \underline{A}) = \left(\frac{\partial}{\partial x_j} \underline{e}_j, A_{ik} \underline{e}_i \otimes \underline{e}_k \right) \\ &= \frac{\partial}{\partial x_j} A_{ik} (\underline{e}_j, \underline{e}_i \otimes \underline{e}_k) \\ &= \frac{\partial A_{ik}}{\partial x_j} \delta_{kj} \underline{e}_i = \frac{\partial A_{ij}}{\partial x_j} \underline{e}_i \end{aligned}$$

The gradient (or derivative) of a tensor \underline{A} is defined as a third order tensor denoted by $\text{grad } \underline{A}$ or $\nabla \otimes \underline{A}$;

$$\text{grad } \underline{A} \text{ or } \nabla \otimes \underline{A} = \frac{\partial}{\partial x_k} \otimes A_{ij} \underline{e}_i \otimes \underline{e}_j \otimes \underline{e}_k$$

We can also define the

$$\begin{aligned} \text{tr}(\text{grad } \underline{A}) &= \text{div } \underline{A} \\ &= \text{grad } \underline{A} : \underline{I} \end{aligned}$$

1.-
Consider scalar functions of one variable,

$$\begin{aligned} \underline{D} &= \underline{D}(t), \quad \underline{u} = \underline{u}(t), \quad \underline{A} = \underline{A}(t) \\ &= u_i(t) \underline{e}_i, \quad = A_{ij}(t) \underline{e}_i \otimes \underline{e}_j \end{aligned}$$

Derivatives: The first derivative \underline{u} and \underline{A} wrt

time is denoted by $\underline{\dot{u}} =: \frac{d}{dt} \underline{u}$

$$\underline{\dot{A}} =: \frac{d}{dt} \underline{A};$$

$$\frac{d}{dt} \underline{e}_i =: 0, \quad \underline{\dot{u}} = \dot{u}_i \underline{e}_i; \quad \underline{\dot{A}} = \dot{A}_{ij} \underline{e}_i \otimes \underline{e}_j$$

Properties; $\frac{d}{dt} (\underline{u} \pm \underline{v}) = \underline{\dot{u}} \pm \underline{\dot{v}}$

$$\frac{d}{dt} \underline{D} \underline{u} = \underline{\dot{D}} \underline{u} + \underline{D} \underline{\dot{u}}$$

$$\frac{d}{dt} (\underline{u} \otimes \underline{v}) = \underline{\dot{u}} \otimes \underline{v} + \underline{u} \otimes \underline{\dot{v}}$$

$$\frac{d}{dt} \text{tr}(\underline{A}) = \text{tr} \underline{\dot{A}}$$

gradient:

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(gradient of a scalar valued (tensor) function:

$$\Phi: \underline{A} \rightarrow \mathbb{R}$$

Aim: approximate the nonlinear function Φ at \underline{A} by a linear function

$$\Phi(\underline{A} + d\underline{A}) \approx \Phi(\underline{A}) + \frac{\partial \Phi}{\partial \underline{A}} : d\underline{A} + o(d\underline{A})$$

$$\left(\frac{\partial \Phi}{\partial \underline{A}} : d\underline{A} \right) = \left(\frac{\partial \Phi}{\partial \underline{A}} ; d\underline{A} \right) \\ = \text{tr} \left(\left(\frac{\partial \Phi}{\partial \underline{A}} \right)^T d\underline{A} \right)$$

$o(\cdot)$ is the Landau order

$$\lim_{d\underline{A} \rightarrow 0} \frac{o(d\underline{A})}{|d\underline{A}|} \rightarrow 0$$

(The term $\frac{\partial \Phi}{\partial \underline{A}} = \text{grad } \Phi(\underline{A})$ denotes the gradient (second order tensor)

Example:

A_{\approx} is square order, invertible.

3-

$$\left(\frac{\partial}{\partial A_{\approx}} \det A_{\approx} = \det A_{\approx} A_{\approx}^{-T} \right)$$

$$\det \begin{pmatrix} A_{\approx} & B_{\approx} \\ C_{\approx} & D_{\approx} \end{pmatrix} = \det(A_{\approx}) \det(D_{\approx})$$

$$\det(A_{\approx} + dA_{\approx}) = \det \left[A_{\approx} \left(\mathbb{I} + A_{\approx}^{-1} dA_{\approx} \right) \right]$$

$$= \det A_{\approx} \det \left(\mathbb{I} + A_{\approx}^{-1} dA_{\approx} \right)$$

$$\det \left(A_{\approx}^{-1} dA_{\approx} + \mathbb{I} \right) = 1 + \text{tr} \left(A_{\approx}^{-1} dA_{\approx} \right) + \mathcal{O}(dA_{\approx})$$

$$\det(A_{\approx} + dA_{\approx}) = \det A_{\approx} + \text{tr} \left[\left(\frac{\partial}{\partial A_{\approx}} \det A_{\approx} \right)^T dA_{\approx} \right] + \mathcal{O}$$

$$\det(A_{\approx} + dA_{\approx}) = \det A_{\approx} \left[1 + \text{tr} \left(A_{\approx}^{-1} dA_{\approx} \right) \right] + \mathcal{O}(dA_{\approx})$$

$$= \det A_{\approx} + \text{tr} \left(\det A_{\approx} A_{\approx}^{-1} dA_{\approx} \right) + \mathcal{O}$$

$$> \frac{\partial}{\partial A_{\approx}} \det A_{\approx} : dA_{\approx} = \left(\det A_{\approx} A_{\approx}^{-1} \right)^T : dA_{\approx}$$

$$= \det A_{\approx} A_{\approx}^{-T} : dA_{\approx}$$

gradient of a tensor-valued function

4.-

Consider $A(\underline{B})$; Compute dA

$$A(\underline{B} + d\underline{B}) = A(\underline{B}) + \frac{\partial A}{\partial \underline{B}} : d\underline{B} + o(d\underline{B})$$

where $\frac{\partial A(\underline{B})}{\partial \underline{B}} = \text{grad}_{\underline{B}} A$

Gradients and Related Operators:

1-

(A tensor $\underline{\underline{A}}$ and a vector \underline{u} , components A_{ij} , u_i depends on Cartesian coordinates -

A tensor field $\underline{\underline{A}}(\underline{x})$, a vector field $\underline{u}(\underline{x})$ -

A scalar field $\Phi(\underline{x})$; $\Phi: \underline{x} \rightarrow \Phi(\underline{x}) \in \mathbb{R}$

gradient of a scalar field:

Taylor expansion Φ for \underline{x} ,

$$(\Phi(\underline{x} + d\underline{x}) = \Phi(\underline{x}) + \frac{\partial \Phi}{\partial \underline{x}} \cdot d\underline{x} + O(d\underline{x}^2))$$

$$(\frac{\partial \Phi}{\partial \underline{x}} \cdot d\underline{x} =: \frac{\partial \Phi}{\partial x_1} dx_1 + \frac{\partial \Phi}{\partial x_2} dx_2 + \frac{\partial \Phi}{\partial x_3} dx_3 \dots)$$

Vector operator ∇ ;

$$\nabla(\cdot) = \frac{\partial}{\partial x_i} \underline{e}_i$$

(The gradient may be written as $(\nabla \Phi, d\underline{x})$)

$$\text{grad } \Phi = \nabla \Phi = \frac{\partial \Phi}{\partial x_i} \underline{e}_i$$

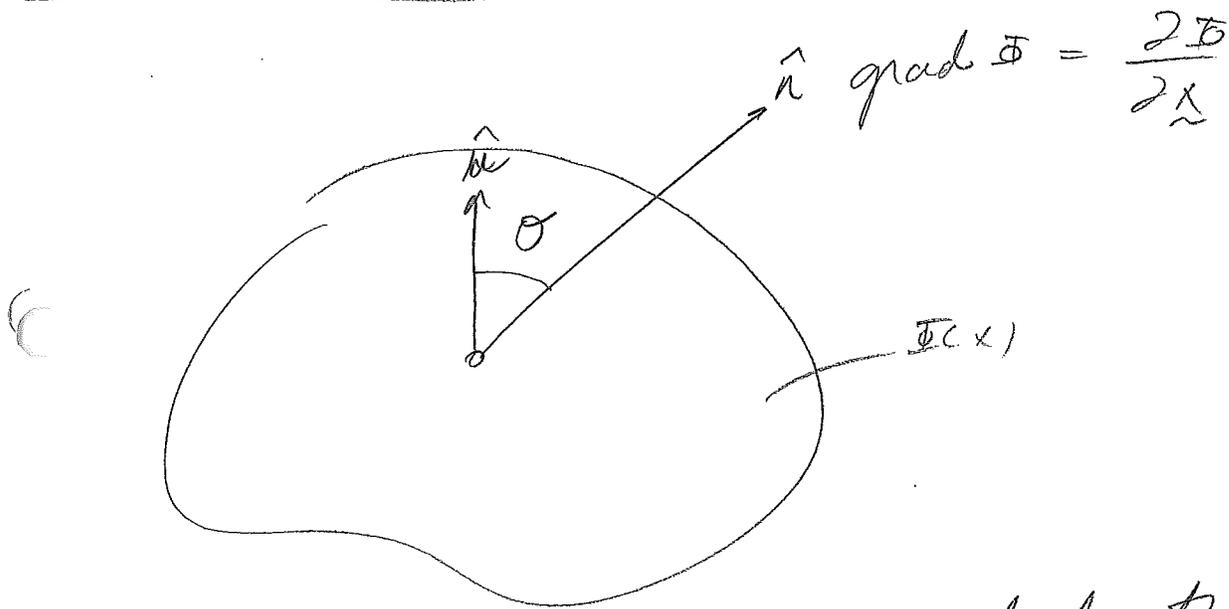
By analogy, other operations may be defined,

2-

$$\nabla \cdot (i) = \left(\frac{\partial}{\partial x_i} (i), \underline{e}_i \right); \quad \nabla \times (i) = \underline{e}_i \times \frac{\partial}{\partial x_i} (i)$$

$$\nabla \otimes (i) = \frac{\partial}{\partial x_i} (i) \otimes \underline{e}_i$$

Directional derivative:



Normal to the surface is described by three values

$$\frac{\partial \Phi}{\partial x_1}, \quad \frac{\partial \Phi}{\partial x_2}, \quad \frac{\partial \Phi}{\partial x_3} \quad (\text{components of grad } \Phi)$$

These are components of a vector perpendicular to the surface of constant Φ

Unit vector normal to surface along Φ , $\hat{n} = \frac{\text{grad } \Phi}{|\text{grad } \Phi|}$

Directional derivative - in twice \underline{u}

3.-

((grad Φ, \underline{u}))

As \underline{u} is varied, (grad Φ, \underline{u}) take a maximum when \underline{u} approaches the direction of grad Φ .

\underline{u} points in the direction of grad Φ ;

Equivalent representation for the directional derivative \underline{u} of Φ ,

(
$$D_{\underline{u}} \Phi(x) = \frac{d}{d\epsilon} \Phi(x + \epsilon \underline{u}) \Big|_{\epsilon=0}$$

$$D_{\underline{u}} \Phi(x) = (\text{grad } \Phi, \underline{u})$$

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Divergence of vector field.

$$\text{div } \underline{u} =: \nabla \cdot \underline{u},$$

the operator applied to \underline{u} (inner product)

$$\begin{aligned} \text{div } \underline{u} &= (\nabla \cdot \underline{u}) = \frac{\partial a_i}{\partial x_i} (\underline{e}_i \cdot \underline{e}_i) \\ &= \frac{\partial a_i}{\partial x_i} \end{aligned}$$

Curl of a vector field:

$$\begin{aligned} \text{curl } \underline{u} &=: \nabla \times \underline{u} = \frac{\partial}{\partial x_i} \underline{e}_i \times \underline{u} \cdot \underline{e}_j \\ &= \frac{\partial a_j}{\partial x_i} \underline{e}_i \times \underline{e}_j \end{aligned}$$

gradient of a vector field: the gradient (or derivative) of a smooth vector field $\underline{u}(\underline{x})$ is a tensor field

$$\text{grad } \underline{u} =: \nabla \otimes \underline{u} = \frac{\partial a_i}{\partial x_j} \underline{e}_i \otimes \underline{e}_j$$

$$\nabla \otimes \nabla = \frac{\partial^2}{\partial x^i \partial x^k} \underline{e}^k \otimes \underline{e}^i$$

$$\text{Tr}(\underline{a} \otimes \underline{b}) = (\nabla \otimes \underline{a}, \underline{b}) = \text{div} \underline{a} = (\nabla, \underline{a})$$

$$\text{tr}(\underline{a} \otimes \underline{b}) = (\underline{a}, \underline{b})$$

Divergence and gradient of 2nd order tensor.

$$\text{Let } \underline{T} = T^i_j \underline{e}_i \otimes \underline{e}^j \quad ; \quad \nabla = \frac{\partial}{\partial x^k} \underline{g}^k$$

$$\begin{aligned} \text{div } \underline{T} &= \nabla T = \frac{\partial}{\partial x^k} T^i_j (\underline{e}_i \otimes \underline{e}^j) \underline{g}^k \\ &= \frac{\partial}{\partial x^k} T^i_j \underline{e}_i (\underline{e}^j \otimes \underline{g}^k) \\ &= T^i_{j,k} \underline{g}^{jk} \underline{e}_i \\ &= T^{ik}{}_{ik} \underline{e}_i \end{aligned}$$

Suppose - $\underline{T} = T_{ij} \underline{e}^i \otimes \underline{e}^j$

$$\begin{aligned} \text{grad } \underline{T} &= \frac{\partial T_{ij}}{\partial x^k} \otimes \underline{g}^k \\ &= \frac{\partial}{\partial x^k} T_{ij} \underline{e}^i \otimes \underline{e}^j \otimes \underline{g}^k \end{aligned}$$

Additional Notes :

1)

Let V, W be vector spaces,

$g: V \rightarrow W$. g is differentiable at x if

$g(x+u) - g(x)$ is equal to a linear function of u plus another term. $\theta \rightarrow 0$ faster than u .

g is differentiable at x

$$\Delta g(x) : U \rightarrow W \quad \exists$$

$$(\Delta g)u = g(x+u) - g(x) + o(u) \quad \text{as } u \rightarrow 0.$$

$\Delta g(x)$ is the derivative of g at x

Ex 1. Let $\varphi(v) = (v, v)$

$$\begin{aligned} \text{Then } (\Delta \varphi(v))u &= \varphi(v+u) - \varphi(v) + o(u) \\ &= (v, v) + 2(v, u) + (u, u) - (v, v) + o(u) \\ &= 2(v, u) \end{aligned}$$

$$\underline{\text{Ex 2.}} - f: L^i \rightarrow L^i$$

(2)

$$f(A) = (t_A)A.$$

By definition:

$$[Df(A)]u = f(A+u) - f(A) + o(A)$$

$$\begin{aligned} f(A+u) &= t_{(A+u)}(A+u) \\ &= (t_A)A + (t_A)u + (t_u)A + (t_u)u \end{aligned}$$

$$f(A+u) - f(A) = (t_A)u + (t_u)A + \overbrace{(t_u)u}^{o(u)}$$

$$Df(A)[u] = (t_A)u + (t_u)A.$$

or

$$Df(A)(u) = \lim_{\alpha \rightarrow 0} \frac{1}{\alpha} \{ f(\alpha + \alpha u) - f(\alpha) \}$$

$$= \frac{d}{d\alpha} f(\alpha + \alpha u) \Big|_{\alpha=0}$$

$$= \frac{d}{d\alpha} \left\{ (t_A)A + \underbrace{(t_A)\alpha u} + \underbrace{(t_{\alpha u})A} + \underbrace{(t_{\alpha u})\alpha u} \right\}$$