Chapter 9

Fanno Flow

9.1 INTRODUCTION

At the start of Chapter 5 we mentioned that area changes, friction, and heat transfer are the most important factors affecting the properties in a flow system. Up to this point we have considered only one of these factors, that of variations in area. However, we have also discussed the various mechanisms by which a flow adjusts to meet imposed boundary conditions of either flow direction or pressure equalization. We now wish to take a look at the subject of friction losses.

To study only the effects of friction, we analyze flow in a constant-area duct without heat transfer. This corresponds to many practical flow situations that involve reasonably short ducts. We consider first the flow of an arbitrary fluid and discover that its behavior follows a definite pattern which is dependent on whether the flow is in the subsonic or supersonic regime. Working equations are developed for the case of a perfect gas, and the introduction of a reference point allows a table to be constructed. As before, the table permits rapid solutions to many problems of this type, which are called *Fanno flow*.

9.2 OBJECTIVES

After completing this chapter successfully, you should be able to:

1. List the assumptions made in the analysis of Fanno flow.
2. (*Optional*) Simplify the general equations of continuity, energy, and momentum to obtain basic relations valid for any fluid in Fanno flow.
3. Sketch a Fanno line in the h–v and the h–s planes. Identify the sonic point and regions of subsonic and supersonic flow.
4. Describe the variation of static and stagnation pressure, static and stagnation temperature, static density, and velocity as flow progresses along a Fanno line. Do for both subsonic and supersonic flow.

5. (*Optional*) Starting with basic principles of continuity, energy, and momentum, derive expressions for property ratios such as T_2/T_1, p_2/p_1, and so on, in terms of Mach number (M) and specific heat ratio (γ) for Fanno flow of a perfect gas.
6. Describe (include T–s diagram) how the Fanno table is developed with the use of a * reference location.
7. Define *friction factor*, *equivalent diameter*, *absolute* and *relative roughness*, *absolute* and *kinematic viscosity*, and *Reynolds number*, and know how to determine each.
8. Compare similarities and differences between Fanno flow and normal shocks. Sketch an h–s diagram showing a typical Fanno line together with a normal shock for the same mass velocity.
9. Explain what is meant by *friction choking*.
10. (*Optional*) Describe some possible consequences of adding duct in a choked Fanno flow situation (for both subsonic and supersonic flow).
11. Demonstrate the ability to solve typical Fanno flow problems by use of the appropriate tables and equations.

9.3 ANALYSIS FOR A GENERAL FLUID

We first consider the general behavior of an arbitrary fluid. To isolate the effects of friction, we make the following assumptions:

Steady one-dimensional flow	
Adiabatic	$\delta q = 0,\ ds_e = 0$
No shaft work	$\delta w_s = 0$
Neglect potential	$dz = 0$
Constant area	$dA = 0$

We proceed by applying the basic concepts of continuity, energy, and momentum.

Continuity

$$\dot{m} = \rho AV = \text{const} \tag{2.30}$$

but since the flow area is constant, this reduces to

$$\rho V = \text{const} \tag{9.1}$$

We assign a new symbol G to this constant (the quantity ρV), which is referred to as the *mass velocity*, and thus

$$\boxed{\rho V = G = \text{const}} \tag{9.2}$$

What are the typical units of G?

Energy

We start with

$$h_{t1} + \cancel{q} = h_{t2} + \cancel{w_s} \tag{3.19}$$

For adiabatic and no work, this becomes

$$h_{t1} = h_{t2} = h_t = \text{const} \tag{9.3}$$

If we neglect the potential term, this means that

$$h_t = h + \frac{V^2}{2g_c} = \text{const} \tag{9.4}$$

Substitute for the velocity from equation (9.2) and *show* that

$$\boxed{h_t = h + \frac{G^2}{\rho^2 2g_c} = \text{const}} \tag{9.5}$$

Now for any given flow, the constant h_t and G are known. Thus equation (9.5) establishes a unique relationship between h and ρ. Figure 9.1 is a plot of this equation in the h–v plane for various values of G (but all for the same h_t). Each curve is called a *Fanno line* and represents flow at a particular *mass velocity*. Note carefully that this is constant G and not constant $\dot{m}$. Ducts of various sizes could pass the same mass flow rate but would have different mass velocities.

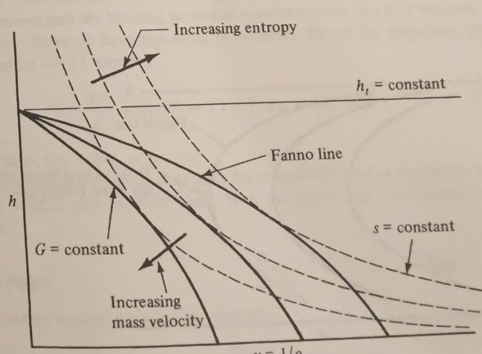

Figure 9.1 Fanno lines in h–v plane.

Once the fluid is known, one can also plot lines of constant entropy on the h diagram. Typical curves of s = constant are shown as dashed lines in the figure is much more instructive to plot these Fanno lines in the familiar h–s plane. Su diagram is shown in Figure 9.2. At this point, a significant fact becomes quite cle Since we have assumed that there is no heat transfer ($ds_e = 0$), the *only* way th entropy can be generated is through irreversibilities (ds_i). Thus *the flow can o progress toward increasing values of entropy*! Why? Can you locate the point o maximum entropy for each Fanno line in Figure 9.1?

Let us examine one Fanno line in greater detail. Figure 9.3 shows a given Fan line together with typical pressure lines. All points on this line represent states wi the same mass flow rate per unit area (mass velocity) and the same stagnation e thalpy. Due to the irreversiible nature of the frictional effects, the flow can only pr ceed to the right. Thus the Fanno line is divided into two distinct parts, an upper a a lower branch, which are separated by a limiting point of maximum entropy.

What does intuition tell us about adiabatic flow in a constant-area duct? We no mally feel that frictional effects will show up as an internal generation of "heat" wi a corresponding reduction in density of the fluid. To pass the same flow rate (wi constant area), continuity then forces the velocity to increase. This increase in kinetic energy must cause a decrease in enthalpy, since the stagnation enthalpy remains con stant. As can be seen in Figure 9.3, this agrees with flow along the *upper branch* of the Fanno line. It is also clear that in this case both the static and stagnation pressure are decreasing.

But what about flow along the *lower branch*? Mark two points on the lower branch and draw an arrow to indicate proper movement along the Fanno line. Wha is happening to the enthalpy? To the density [see equation (9.5)]? To the velocity [see equation (9.2)]? From the figure, what is happening to the static pressure? The stagnation pressure? Fill in Table 9.1 with *increase*, *decrease*, or *remains constant*.

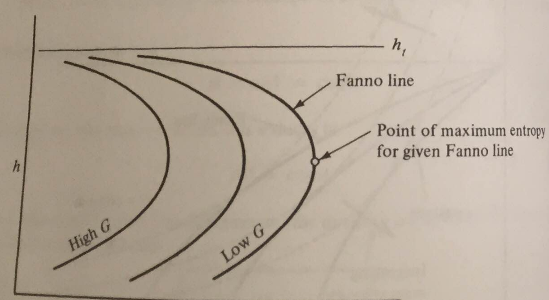

Figure 9.2 Fanno lines in h–s plane.

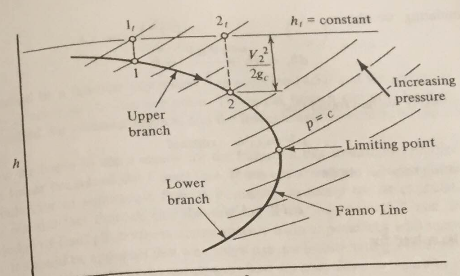

Figure 9.3 Two branches of a Fanno line.

Table 9.1 Analysis of Fanno Flow for Figure 9.3

Property	Upper Branch	Lower Branch
Enthalpy		
Density		
Velocity		
Pressure (static)		
Pressure (stagnation)		

Notice that on the lower branch, properties do not vary in the manner predicted by *intuition*. Thus this must be a flow regime with which we are not very familiar. Before we investigate the limiting point that separates these two flow regimes, let us note that these flows do have one thing in common. Recall the stagnation pressure energy equation from Chapter 3.

$$\frac{dp_t}{\rho_t} + \cancel{d}s_e(T_t - T) + T_t\,ds_i + \delta\cancel{w}_s = 0 \tag{3.25}$$

For Fanno flow, $ds_e = \delta w_s = 0$.

Thus any frictional effect must cause a decrease in the total or stagnation pressure! Figure 9.3 verifies this for flow along both the upper and lower branches of the Fanno line.

Limiting Point

From the energy equation we had developed,

$$h_t = h + \frac{V^2}{2g_c} = \text{constant} \tag{9.4}$$

Differentiating, we obtain

$$dh_t = dh + \frac{V\,dV}{g_c} = 0 \tag{9.6}$$

From continuity we had found that

$$\rho V = G = \text{constant} \tag{9.2}$$

Differentiating this, we obtain

$$\rho\,dV + V\,d\rho = 0 \tag{9.7}$$

which can be solved for

$$dV = -V\frac{d\rho}{\rho} \tag{9.8}$$

Introduce equation (9.8) into (9.6) and *show* that

$$dh = \frac{V^2\,d\rho}{g_c\rho} \tag{9.9}$$

Now recall the property relation

$$T\,ds = dh - v\,dp \tag{1.41}$$

which can be written as

$$T\,ds = dh - \frac{dp}{\rho} \tag{9.10}$$

Substituting for dh from equation (9.9) yields

$$\boxed{T\,ds = \frac{V^2\,d\rho}{g_c\rho} - \frac{dp}{\rho}} \tag{9.11}$$

We hasten to point out that this expression is valid for *any* fluid and between two differentially separated points *anyplace* along the Fanno line. Now let's apply equation (9.11) to two adjacent points that surround the limiting point of maximum entropy. At this location $s =$ const; thus $ds = 0$, and (9.11) becomes

$$\frac{V^2\,d\rho}{g_c} = dp \quad \text{at limit point} \tag{9.12}$$

or

$$V^2 = g_c\left(\frac{dp}{d\rho}\right)_{\text{at limit point}} = g_c\left(\frac{\partial p}{\partial \rho}\right)_{s\,=\,\text{const}} \tag{9.13}$$

This should be a familiar expression [see equation (4.5)] and we recognize that *the velocity is sonic at the limiting point*. The upper branch can now be more significantly called the *subsonic branch*, and the lower branch is seen to be the *supersonic branch*.

Now we begin to see a reason for the failure of our intuition to predict behavior on the lower branch of the Fanno line. From our studies in Chapter 5 we saw that fluid behavior in supersonic flow is frequently contrary to our expectations. This points out the fact that we live most of our lives "subsonically," and, in fact, our knowledge of fluid phenomena comes mainly from experiences with incompressible fluids. It should be apparent that we cannot use our intuition to guess at what might be happening, particularly in the supersonic flow regime. We must learn to get religious and put faith in our carefully derived relations.

Momentum

The foregoing analysis was made using only the continuity and energy relations. We now proceed to apply momentum concepts to the control volume shown in Figure 9.4. The x-component of the momentum equation for steady, one-dimensional flow is

$$\sum F_x = \frac{\dot{m}}{g_c}\left(V_{\text{out}_x} - V_{\text{in}_x}\right) \tag{3.46}$$

From Figure 9.4 we see that the force summation is

$$\sum F_x = p_1 A - p_2 A - F_f \tag{9.14}$$

where F_f represents the total wall frictional force on the fluid between sections 1 and 2. Thus the momentum equation in the direction of flow becomes

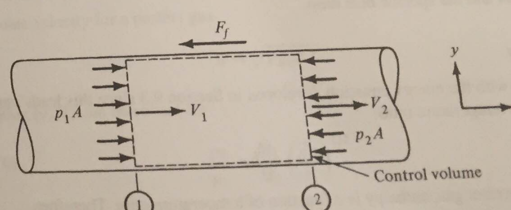

Figure 9.4 Momentum analysis for Fanno flow.

$$(p_1 - p_2)A - F_f = \frac{\dot{m}}{g_c}(V_2 - V_1) = \frac{\rho AV}{g_c}(V_2 - V_1) \qquad (9.15)$$

Show that equation (9.15) can be written as

$$p_1 - p_2 - \frac{F_f}{A} = \frac{\rho_2 V_2^2}{g_c} - \frac{\rho_1 V_1^2}{g_c} \qquad (9.16)$$

or

$$\boxed{\left(p_1 + \frac{\rho_1 V_1^2}{g_c}\right) - \frac{F_f}{A} = p_2 + \frac{\rho_2 V_2^2}{g_c}} \qquad (9.17)$$

In this form the equation is not particularly useful except to bring out one significant fact. *For the steady, one-dimensional, constant-area flow of any fluid*, the value of $p + \rho V^2/g_c$ *cannot* be constant if frictional forces are present. This fact will be recalled later in the chapter when Fanno flow is compared with normal shocks.

Before leaving this section on fluids in general, we might say a few words about Fanno flow at low Mach numbers. A glance at Figure 9.3 shows that the upper branch is asymtotically approaching the horizontal line of constant total enthalpy. Thus the extreme left end of the Fanno line will be nearly horizontal. This indicates that flow at very low Mach numbers will have almost constant velocity. This checks our previous work, which indicated that we could treat gases as incompressible fluids if the Mach numbers were very small.

9.4 WORKING EQUATIONS FOR PERFECT GASES

We have discovered the general trend of property variations that occur in Fanno flow both in the subsonic and supersonic flow regime. Now we wish to develop some specific working equations for the case of a perfect gas. Recall that these are relations between properties at arbitrary sections of a flow system written in terms of Mach numbers and the specific heat ratio.

Energy

We start with the energy equation developed in Section 9.3 since this leads immediately to a temperature ratio:

$$h_{t1} = h_{t2} \qquad (9.3)$$

But for a perfect gas, enthalpy is a function of temperature only. Therefore,

$$T_{t1} = T_{t2} \qquad (9.18)$$

Now for a perfect gas with constant specific heats,

$$T_t = T\left(1 + \frac{\gamma - 1}{2}M^2\right) \qquad (4.18)$$

Hence the energy equation for Fanno flow can be written as

$$T_1\left(1 + \frac{\gamma - 1}{2}M_1^2\right) = T_2\left(1 + \frac{\gamma - 1}{2}M_2^2\right) \qquad (9.19)$$

or

$$\boxed{\frac{T_2}{T_1} = \frac{1 + [(\gamma - 1)/2]M_1^2}{1 + [(\gamma - 1)/2]M_2^2}} \qquad (9.20)$$

Continuity

From Section 9.3 we have

$$\rho V = G = \text{const} \qquad (9.2)$$

or

$$\rho_1 V_1 = \rho_2 V_2 \qquad (9.21)$$

If we introduce the perfect gas equation of state

$$p = \rho RT \qquad (1.13)$$

the definition of Mach number

$$V = Ma \qquad (4.11)$$

and sonic velocity for a perfect gas

$$a = \sqrt{\gamma g_c RT} \qquad (4.10)$$

equation (9.21) can be solved for

$$\frac{p_2}{p_1} = \frac{M_1}{M_2}\left(\frac{T_2}{T_1}\right)^{1/2} \qquad (9.22)$$

Can you obtain this expression? Now introduce the temperature ratio from (9.20) and you will have the following working relation for static pressure:

$$\boxed{\frac{p_2}{p_1} = \frac{M_1}{M_2}\left(\frac{1+[(\gamma-1)/2]M_1^2}{1+[(\gamma-1)/2]M_2^2}\right)^{1/2}} \qquad (9.23)$$

The density relation can easily be obtained from equation (9.20), (9.23), and the perfect gas law:

$$\boxed{\frac{\rho_2}{\rho_1} = \frac{M_1}{M_2}\left(\frac{1+[(\gamma-1)/2]M_2^2}{1+[(\gamma-1)/2]M_1^2}\right)^{1/2}} \qquad (9.24)$$

Entropy Change

We start with an expression for entropy change that is valid between any two points:

$$\Delta s_{1-2} = c_p \ln\frac{T_2}{T_1} - R\ln\frac{p_2}{p_1} \qquad (1.53)$$

Equation (4.15) can be used to substitute for c_p and we nondimensionalize the equation to

$$\frac{s_2 - s_1}{R} = \frac{\gamma}{\gamma-1}\ln\frac{T_2}{T_1} - \ln\frac{p_2}{p_1} \qquad (9.25)$$

If we now utilize the expressions just developed for the temperature ratio (9.20) and the pressure ratio (9.23), the entropy change becomes

$$\frac{s_2 - s_1}{R} = \frac{\gamma}{\gamma-1}\ln\left(\frac{1+[(\gamma-1)/2]M_1^2}{1+[(\gamma-1)/2]M_2^2}\right) - \ln\frac{M_1}{M_2}\left(\frac{1+[(\gamma-1)/2]M_1^2}{1+[(\gamma-1)/2]M_2^2}\right)^{1/2} \qquad (9.26)$$

Show that this entropy change between two points in Fanno flow can be written as

$$\frac{s_2 - s_1}{R} = \ln\frac{M_2}{M_1}\left(\frac{1+[(\gamma-1)/2]M_1^2}{1+[(\gamma-1)/2]M_2^2}\right)^{(\gamma+1)/2(\gamma-1)} \qquad (9.27)$$

Now recall that in Section 4.5 we integrated the stagnation pressure–energy equation for adiabatic no-work flow of a perfect gas, with the result

$$\frac{p_{t2}}{p_{t1}} = e^{-\Delta s/R} \qquad (4.28)$$

Thus, from equations (4.28) and (9.27) we obtain a simple expression for the stagnation pressure ratio:

$$\boxed{\frac{p_{t2}}{p_{t1}} = \frac{M_1}{M_2}\left(\frac{1+[(\gamma-1)/2]M_2^2}{1+[(\gamma-1)/2]M_1^2}\right)^{(\gamma+1)/2(\gamma-1)}} \qquad (9.28)$$

We now have the means to obtain all the properties at a downstream point 2 if we know all the properties at some upstream point 1 *and* the Mach number at point 2. However, in many situations one does not know both Mach numbers. A typical problem would be to predict the final Mach number, given the initial conditions and information on duct length, material, and so on. Thus our next job is to relate the change in Mach number to the friction losses.

Momentum

We turn to the differential form of the momentum equation that was developed in Chapter 3:

$$\frac{dp}{\rho} + f\frac{V^2\,dx}{2g_cD_e} + \frac{g}{g_c}\,dz + \frac{dV^2}{2g_c} = 0 \qquad (3.63)$$

Our objective is to get this equation all in terms of Mach number. If we introduce the perfect gas equation of state together with expressions for Mach number and sonic velocity, we obtain

$$\frac{dp}{p}(RT) + f\frac{dx}{D_e}\frac{M^2\gamma g_c RT}{2g_c} + \frac{g}{g_c}\,dz + \frac{dM^2\gamma g_c RT + M^2\gamma g_c R\,dT}{2g_c} = 0 \qquad (9.29)$$

or

$$\boxed{\frac{dp}{p} + f\frac{dx}{D_e}\frac{\gamma}{2}M^2 + \frac{g\,dz}{g_c RT} + \frac{\gamma}{2}dM^2 + \frac{\gamma}{2}M^2\frac{dT}{T} = 0} \qquad (9.30)$$

Equation (9.30) is boxed since it is a useful form of the momentum equation that is valid for *all* steady flow problems involving a perfect gas. We now proceed to apply this to Fanno flow. From (9.18) and (4.18) we know that

$$T_t = T\left(1+\frac{\gamma-1}{2}M^2\right) = \text{const} \qquad (9.31)$$

Taking the natural logarithm

$$\ln T + \ln\left(1 + \frac{\gamma - 1}{2} M^2\right) = \ln \text{const}$$

and then differentiating, we obtain

$$\frac{dT}{T} + \frac{d(1 + [(\gamma - 1)/2]M^2)}{1 + [(\gamma - 1)/2]M^2} = 0 \tag{9.33}$$

which can be used to substitute for dT/T in (9.30).

The continuity relation [equation (9.2)] put in terms of a perfect gas becomes

$$\frac{pM}{\sqrt{T}} = \text{const} \tag{9.34}$$

By logarithmic differentiation (take the natural logarithm and then differentiate) show that

$$\frac{dp}{p} + \frac{dM}{M} - \frac{1}{2}\frac{dT}{T} = 0 \tag{9.35}$$

We can introduce equation (9.33) to eliminate dT/T, with the result that

$$\frac{dp}{p} = -\frac{dM}{M} - \frac{1}{2}\frac{d(1 + [(\gamma - 1)/2]M^2)}{1 + [(\gamma - 1)/2]M^2} \tag{9.36}$$

which can be used to substitute for dp/p in (9.30).

Make the indicated substitutions for dp/p and dT/T in the momentum equation, neglect the potential term, and *show* that equation (9.30) can be put into the following form:

$$f\frac{dx}{D_e} = \frac{d(1 + [(\gamma - 1)/2]M^2)}{1 + [(\gamma - 1)/2]M^2} - \frac{dM^2}{M^2} + \frac{2}{\gamma}\frac{dM}{M^3} + \frac{1}{\gamma M^2}\frac{d(1 + [(\gamma - 1)/2]M^2)}{1 + [(\gamma - 1)/2]M^2} \tag{9.37}$$

The last term can be simplified for integration by noting that

$$\frac{1}{\gamma M^2}\frac{d(1 + [(\gamma - 1)/2]M^2)}{1 + [(\gamma - 1)/2]M^2} = \frac{(\gamma - 1)}{2\gamma}\frac{dM^2}{M^2} - \frac{(\gamma - 1)}{2\gamma}\frac{d(1 + [(\gamma - 1)/2]M^2)}{1 + [(\gamma - 1)/2]M^2} \tag{9.38}$$

The momentum equation can now be written as

$$f\frac{dx}{D_e} = \frac{\gamma + 1}{2\gamma}\frac{d(1 + [(\gamma - 1)/2]M^2)}{1 + [(\gamma - 1)/2]M^2} + \frac{2}{\gamma}\frac{dM}{M^3} - \frac{\gamma + 1}{2\gamma}\frac{dM^2}{M^2} \tag{9.39}$$

Equation (9.39) is restricted to steady, one-dimensional flow of a perfect gas, with no heat or work transfer, constant area, and negligible potential changes. We can now integrate this equation between two points in the flow and obtain

$$\boxed{\frac{f(x_2 - x_1)}{D_e} = \frac{\gamma + 1}{2\gamma}\ln\frac{1 + [(\gamma - 1)/2]M_2^2}{1 + [(\gamma - 1)/2]M_1^2} - \frac{1}{\gamma}\left(\frac{1}{M_2^2} - \frac{1}{M_1^2}\right) - \frac{\gamma + 1}{2\gamma}\ln\frac{M_2^2}{M_1^2}} \tag{9.40}$$

Note that in performing the integration we have held the friction factor constant. Some comments will be made on this in a later section. If you have forgotten the concept of equivalent diameter, you may want to review the last part of Section 3.8 and equation (3.61).

9.5 REFERENCE STATE AND FANNO TABLE

The equations developed in Section 9.4 provide the means of computing the properties at one location in terms of those given at some other location. The key to problem solution is predicting the Mach number at the new location through the use of equation (9.40). The solution of this equation for the unknown M_2 presents a messy task, as no explicit relation is possible. Thus we turn to a technique similar to that used with isentropic flow in Chapter 5.

We introduce *another* * reference state, which is defined in the same manner as before (i.e., "that thermodynamic state which would exist if the fluid reached a Mach number of unity *by a particular process*"). In this case we imagine that we continue *by Fanno flow* (i.e., more duct is added) until the velocity reaches Mach 1. Figure 9.5 shows a physical system together with its T–s diagram for a subsonic Fanno flow. We know that if we continue along the Fanno line (remember that we always move to the right), we will eventually reach the limiting point where sonic velocity exists. The dashed lines show a hypothetical duct of sufficient length to enable the flow to traverse the remaining portion of the upper branch and reach the limit point. This is the * reference point *for Fanno flow*.

The *isentropic* * reference points have also been included on the T–s diagram to emphasize the fact that the Fanno * reference is a totally different thermodynamic state. One other fact should be mentioned. If there is any entropy difference between two points (such as points 1 and 2), their isentropic * reference conditions are not the same and we have always taken great care to label them separately as 1^* and 2^*.

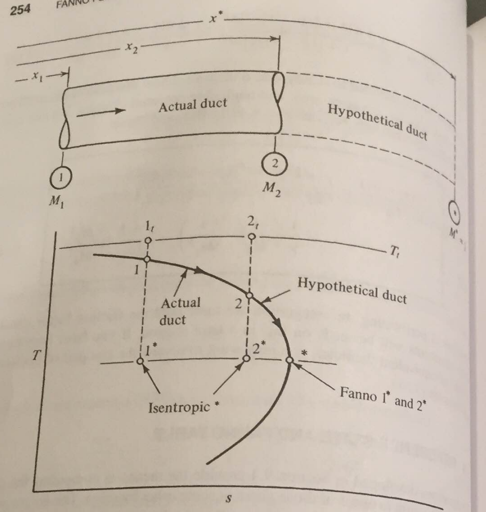

Figure 9.5 The * reference for Fanno flow.

However, proceeding from either point 1 or point 2 *by Fanno flow* will ultimately lead to the same place when Mach 1 is reached. Thus we do not have to talk of 1* or 2* but merely * in the case of Fanno flow. Incidentally, why are all three * reference points shown on the same horizontal line in Figure 9.5? (You may need to review Section 4.6.)

We now rewrite the working equations in terms of the Fanno flow * reference condition. Consider first

$$\frac{T_2}{T_1} = \frac{1 + [(\gamma - 1)/2]M_1^2}{1 + [(\gamma - 1)/2]M_2^2} \tag{9.20}$$

Let point 2 be an arbitrary point in the flow system and let its Fanno * condition be point 1. Then

$$T_2 \Rightarrow T \qquad M_2 \Rightarrow M \quad \text{(any value)}$$
$$T_1 \Rightarrow T^* \qquad M_1 \Rightarrow 1$$

and equation (9.20) becomes

$$\frac{T}{T^*} = \frac{(\gamma + 1)/2}{1 + [(\gamma - 1)/2]M^2} = f(M, \gamma) \tag{9.41}$$

We see that $T/T^* = f(M, \gamma)$ and we can easily construct a table giving values of T/T^* versus M for a particular γ. Equation (9.23) can be treated in a similar fashion. In this case

$$p_2 \Rightarrow p \qquad M_2 \Rightarrow M \quad \text{(any value)}$$
$$p_1 \Rightarrow p^* \qquad M_1 \Rightarrow 1$$

and equation (9.23) becomes

$$\frac{p}{p^*} = \frac{1}{M}\left(\frac{(\gamma + 1)/2}{1 + [(\gamma - 1)/2]M^2}\right)^{1/2} = f(M, \gamma) \tag{9.42}$$

The density ratio can be obtained as a function of Mach number and γ from equation (9.24). This is particularly useful since it also represents a velocity ratio. Why?

$$\frac{\rho}{\rho^*} = \frac{V^*}{V} = \frac{1}{M}\left(\frac{1 + [(\gamma - 1)/2]M^2}{(\gamma + 1)/2}\right)^{1/2} = f(M, \gamma) \tag{9.43}$$

Apply the same techniques to equation (9.28) and *show* that

$$\frac{p_t}{p_t^*} = \frac{1}{M}\left(\frac{1 + [(\gamma - 1)/2]M^2}{(\gamma + 1)/2}\right)^{(\gamma+1)/2(\gamma-1)} = f(M, \gamma) \tag{9.44}$$

We now perform the same type of transformation on equation (9.40); that is, let

$$x_2 \Rightarrow x \qquad M_2 \Rightarrow M \quad \text{(any value)}$$
$$x_1 \Rightarrow x^* \qquad M_1 \Rightarrow 1$$

with the following result:

$$\frac{f(x - x^*)}{D_e} = \left(\frac{\gamma + 1}{2\gamma}\right) \ln\left(\frac{1 + [(\gamma - 1)/2]M^2}{(\gamma + 1)/2}\right) - \frac{1}{\gamma}\left(\frac{1}{M^2} - 1\right) - \frac{\gamma + 1}{2\gamma} \ln M^2 \tag{9.45}$$

But a glance at the physical diagram in Figure 9.5 shows that $(x^* - x)$ will always be a negative quantity; thus it is more convenient to change all signs in equation (9.45) and simplify it to

$$\frac{f(x^* - x)}{D_e} = \left(\frac{\gamma + 1}{2\gamma}\right) \ln\left(\frac{[(\gamma + 1)/2]M^2}{1 + [(\gamma - 1)/2]M^2}\right) + \frac{1}{\gamma}\left(\frac{1}{M^2} - 1\right) = f(M, \gamma) \tag{9.46}$$

The quantity $(x^* - x)$ represents the amount of duct that would have to be added to cause the flow to reach the Fanno * reference condition. It can alternatively be viewed as the maximum duct length that may be added without changing some flow condition. Thus the expression

$$\frac{f(x^* - x)}{D_e} \quad \text{is called} \quad \frac{f L_{max}}{D_e}$$

and is listed in Appendix I along with the other Fanno flow parameters: T/T^*, p/p^*, V/V^*, and p_t/p_t^*. In the next section we shall see how this table greatly simplifies the solution of Fanno flow problems. But first, some words about the determination of friction factors.

Dimensional analysis of the fluid flow problem shows that the friction factor can be expressed as

$$f = f(\text{Re}, \varepsilon/D) \tag{9.47}$$

where Re is the *Reynolds number*,

$$\text{Re} \equiv \frac{\rho V D}{\mu g_c} \tag{9.48}$$

and

$$\varepsilon/D \equiv \textit{relative roughness}$$

Typical values of ε, the *absolute roughness* or average height of wall irregularities, are shown in Table 9.2.

The relationship among f, Re, and ε/D is determined experimentally and plotted on a chart similar to Figure 9.6, which is called a *Moody diagram*. A larger working chart appears in Appendix C. If the flow rate is known together with the duct size and

Table 9.2 Absolute Roughness of Common Materials

Material	ε (ft)
Glass, brass, copper, lead	smooth < 0.00001
Steel, wrought iron	0.00015
Galvanized iron	0.0005
Cast iron	0.00085
Riveted steel	0.03

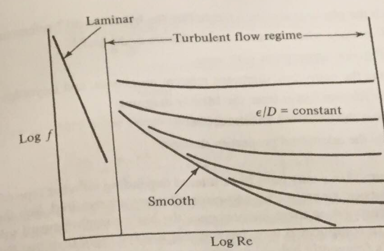

Figure 9.6 Moody diagram for friction factor in circular ducts. (See Appendix C for working chart.)

material, the Reynolds number and relative roughness can easily be calculated and the value of the friction factor is taken from the diagram. The curve in the laminar flow region can be represented by

$$f = \frac{64}{\text{Re}} \tag{9.49}$$

For noncircular cross sections the *equivalent diameter* as described in Section 3.8 can be used.

$$D_e \equiv \frac{4A}{P} \tag{3.61}$$

This equivalent diameter may be used in the determination of relative roughness and Reynolds number, and hence the friction factor. However, care must be taken to work with the *actual* average velocity in all computations. Experience has shown that the use of an equivalent diameter works quite well in the turbulent zone. In the laminar flow region this concept is not sufficient and consideration must also be given to the aspect ratio of the duct.

In some problems the flow rate is not known and thus a trial-and-error solution results. As long as the duct size is given, the problem is not too difficult; an excellent approximation to the friction factor can be made by taking the value corresponding to where the ε/D curve begins to *level off*. This converges rapidly to the final answer, as most engineering problems are well into the turbulent range.

9.6 APPLICATIONS

The following steps are recommended to develop good problem-solving technique:

1. Sketch the physical situation (including the hypothetical * reference poin
2. Label sections where conditions are known or desired.
3. List all given information with units.
4. Compute the equivalent diameter, relative roughness, and Reynolds numbe
5. Find the friction factor from the Moody diagram.
6. Determine the unknown Mach number.
7. Calculate the additional properties desired.

The procedure above may have to be altered depending on what type of infor tion is given, and occasionally, trial-and-error solutions are required. You should ha no difficulty incorporating these features once the basic straightforward solution ha been mastered. In complicated flow systems that involve more than just Fanno flo a T–s diagram is frequently helpful in solving problems.

For the following examples we are dealing with the steady one-dimensional flo of air ($\gamma = 1.4$), which can be treated as a perfect gas. Assume that $Q = W_s = 0$ an negligible potential changes. The cross-sectional area of the duct remains constan Figure E9.1 is common to Examples 9.1 through 9.3.

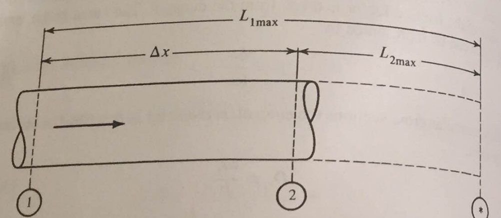

Figure E9.1

Example 9.1 Given $M_1 = 1.80$, $p_1 = 40$ psia, and $M_2 = 1.20$, find p_2 and $f\Delta x/D$.
Since both Mach numbers are known, we can solve immediately for

$$p_2 = \frac{p_2}{p^*}\frac{p^*}{p_1}p_1 = (0.8044)\left(\frac{1}{0.4741}\right)(40) = 67.9 \text{ psia}$$

Check Figure E9.1 to see that

$$\frac{f\Delta x}{D} = \frac{fL_{1\max}}{D} - \frac{fL_{2\max}}{D} = 0.2419 - 0.0336 = 0.208$$

Example 9.2 Given $M_2 = 0.94$, $T_1 = 400$ K, and $T_2 = 350$ K, find M_1 and p_2/p_1.
To determine conditions at section 1 in Figure E9.1, we must establish the ratio

$$\frac{T_1}{T^*} = \frac{T_1}{T_2}\frac{T_2}{T^*} = \left(\frac{400}{350}\right)(1.0198) = 1.1655$$

From Fanno table at $M = 0.94$

Given

Look up $T/T^* = 1.1655$ in the Fanno table (Appendix I) and determine that $M_1 = 0.385$.
Thus

$$\frac{p_2}{p_1} = \frac{p_2}{p^*}\frac{p^*}{p_1} = (1.0743)\left(\frac{1}{2.8046}\right) = 0.383$$

Notice that these examples confirm previous statements concerning static pressure changes. In subsonic flow the static pressure decreases, whereas in supersonic flow the static pressure increases. *Compute* the stagnation pressure ratio and show that the friction losses cause p_{t2}/p_{t1} to decrease in each case.

For Example 9.1:

$$\frac{p_{t2}}{p_{t1}} = \qquad (p_{t2}/p_{t1} = 0.716)$$

For Example 9.2:

$$\frac{p_{t2}}{p_{t1}} = \qquad (p_{t2}/p_{t1} = 0.611)$$

Example 9.3 Air flows in a 6-in.-diameter, insulated, galvanized iron duct. Initial conditions are $p_1 = 20$ psia, $T_1 = 70$°F, and $V_1 = 406$ ft/sec. After 70 ft, determine the final Mach number, temperature, and pressure.

Since the duct is circular we do not have to compute an equivalent diameter. From Table 9.2 the absolute roughness ε is 0.0005. Thus the relative roughness

$$\frac{\varepsilon}{D} = \frac{0.0005}{0.5} = 0.001$$

We compute the Reynolds number at section 1 (Figure E9.1) since this is the only location where information is known.

$$\rho_1 = \frac{p_1}{RT_1} = \frac{(20)(144)}{(53.3)(530)} = 0.102 \text{ lbm/ft}^3$$

$$\mu_1 = 3.8 \times 10^{-7} \text{ lbf-sec/ft}^2 \text{ (from table in Appendix A)}$$

Thus

$$\text{Re}_1 = \frac{\rho_1 V_1 D_1}{\mu_1 g_c} = \frac{(0.102)(406)(0.5)}{(3.8 \times 10^{-7})(32.2)} = 1.69 \times 10^6$$

From the Moody diagram (in Appendix C) at Re $= 1.69 \times 10^6$ and $\varepsilon/D = 0.001$, we determine that the friction factor is $f = 0.0198$. To use the Fanno table (or equations), we need information on Mach numbers.

$$a_1 = (\gamma g_c R T_1)^{1/2} = [(1.4)(32.2)(53.3)(530)]^{1/2} = 1128 \text{ ft/sec}$$

$$M_1 = \frac{V_1}{a_1} = \frac{406}{1128} = 0.36$$

From the Fanno table (Appendix I) at $M_1 = 0.36$, we find that

$$\frac{p_1}{p^*} = 3.0042 \qquad \frac{T_1}{T^*} = 1.1697 \qquad \frac{fL_{1\,\max}}{D} = 3.1801$$

The key to completing the problem is in establishing the Mach number at the outlet, and this is done through the *friction length*:

$$\frac{f\,\Delta x}{D} = \frac{(0.0198)(70)}{0.5} = 2.772$$

Looking at the physical sketch it is apparent (since f and D are constants) that

$$\frac{fL_{2\,\max}}{D} = \frac{fL_{1\,\max}}{D} - \frac{f\,\Delta x}{D} = 3.1801 - 2.772 = 0.408$$

We enter the Fanno table with this friction length and find that

$$M_2 = 0.623 \qquad \frac{p_2}{p^*} = 1.6939 \qquad \frac{T_2}{T^*} = 1.1136$$

Thus

$$p_2 = \frac{p_2}{p^*}\frac{p^*}{p_1}p_1 = (1.6939)\left(\frac{1}{3.0042}\right)(20) = 11.28 \text{ psia}$$

and

$$T_2 = \frac{T_2}{T^*}\frac{T^*}{T_1}T_1 = (1.1136)\left(\frac{1}{1.1697}\right)(530) = 505°\text{R}$$

In the example above, the friction factor was assumed constant. In fact, this assumption was made when equation (9.39) was integrated to obtain (9.40), and with the introduction of the * reference state, this became equation (9.46), which is listed in the Fanno table. Is this a reasonable assumption? Friction factors are functions of Reynolds numbers, which in turn depend on velocity and density—both of which can change quite rapidly in Fanno flow. Calculate the velocity at the outlet in Example 9.3 and compare it with that at the inlet. ($V_2 = 686$ ft/sec and $V_1 = 406$ ft/sec.)

But don't despair. From continuity we know that the product of ρV is always constant, and thus the only variable in Reynolds number is the viscosity. Extremely large temperature variations are required to change the viscosity of a gas significantly and thus variations in the Reynolds number are small for any given problem. We are also fortunate in that most engineering problems are well into the turbulent range where the friction factor is relatively insensitive to Reynolds number. A greater potential error is involved in the estimation of the duct roughness, which has a more significant effect on the friction factor.

Example 9.4 A converging–diverging nozzle with an area ratio of 5.42 connects to an 8-ft-long constant-area rectangular duct (see Figure E9.4). The duct is 8 × 4 in. in cross section and has a friction factor of $f = 0.02$. What is the minimum stagnation pressure feeding the nozzle if the flow is supersonic throughout the entire duct and it exhausts to 14.7 psia?

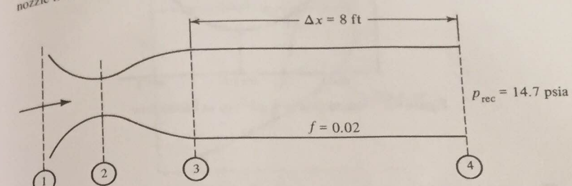

Figure E9.4

$$D_e = \frac{4A}{P} = \frac{(4)(32)}{24} = 5.334 \text{ in.}$$

$$\frac{f\,\Delta x}{D} = \frac{(0.02)(8)(12)}{5.334} = 0.36$$

To be supersonic with $A_3/A_2 = 5.42$, $M_3 = 3.26$, $p_3/p_{t3} = 0.0185$, $p_3/p^* = 0.1901$, and $fL_{3\max}/D = 0.5582$,

$$\frac{fL_{4\,\max}}{D} = \frac{fL_{3\,\max}}{D} - \frac{f\,\Delta x}{D} = 0.5582 - 0.36 = 0.1982$$

Thus

$$M_4 = 1.673 \quad \text{and} \quad \frac{p_4}{p^*} = 0.5243$$

and

$$p_{t1} = \frac{p_{t1}}{p_{t3}}\frac{p_{t3}}{p_3}\frac{p_3}{p^*}\frac{p^*}{p_4}p_4 = (1)\left(\frac{1}{0.0185}\right)(0.1901)\left(\frac{1}{0.5243}\right)(14.7) = 228 \text{ psia}$$

Any pressure above 288 psia will maintain the flow system as specified but with expansion waves outside the duct. (Recall an underexpanded nozzle.) Can you envision what would happen if the inlet stagnation pressure fell below 288 psia? (Recall the operation of an overexpanded nozzle.)

9.7 CORRELATION WITH SHOCKS

As you have progressed through this chapter you may have noticed some similarities between Fanno flow and normal shocks. Let us summarize some pertinent information.

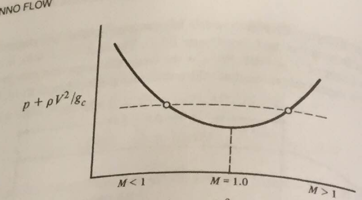

Figure 9.7 Variation of $p + \rho V^2/g_c$ in Fanno flow.

The points just before and after a normal shock represent states with the same mass flow per unit area, the same value of $p + \rho V^2/g_c$, and the same stagnation enthalpy. These facts are the result of applying the basic concepts of continuity, momentum, and energy to any arbitrary fluid. This analysis resulted in equations (6.2), (6.3), and (6.9).

A Fanno line represents states with the same mass flow per unit area and the same stagnation enthalpy. This is confirmed by equations (9.2) and (9.5). To move along a Fanno line requires friction. At the end of Section 9.3 [see equation (9.17)] it was pointed out that it is this very friction which causes the value of $p + \rho V^2/g_c$ to change.

The variation of the quantity $p + \rho V^2/g_c$ along a Fanno line is quite interesting. Such a plot is shown in Figure 9.7. You will notice that for every point on the supersonic branch of the Fanno line there is a corresponding point on the subsonic branch with the same value of $p + \rho V^2/g_c$. Thus these two points satisfy all three conditions for the end points of a normal shock and could be connected by such a shock.

Now we can imagine a supersonic Fanno flow leading into a normal shock. If this is followed by additional duct, subsonic Fanno flow would occur. Such a situation is shown in Figure 9.8*a*. Note that the shock merely causes the flow to jump from the supersonic branch to the subsonic branch of the *same* Fanno line. [See Figure 9.8*b*.]

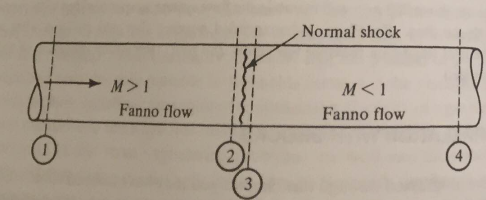

Figure 9.8*a* Combination of Fanno flow and normal shock (physical system).

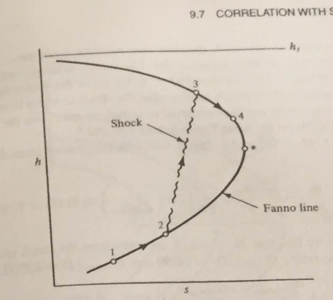

Figure 9.8*b* Combination of Fanno flow and normal shock.

Example 9.5 A large chamber contains air at a temperature of 300 K and a pressure of 8 bar abs (Figure E9.5). The air enters a converging–diverging nozzle with an area ratio of 2.4. A constant-area duct is attached to the nozzle and a normal shock stands at the exit plane. Receiver pressure is 3 bar abs. Assume the entire system to be adiabatic and neglect friction in the nozzle. Compute the $f\,\Delta x/D$ for the duct.

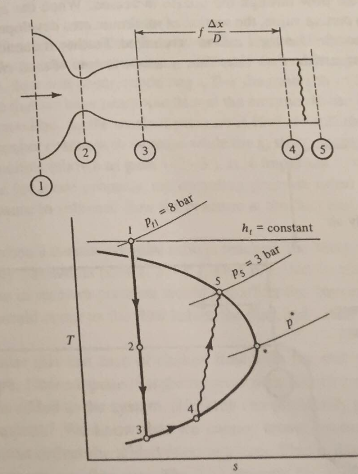

Figure E9.5

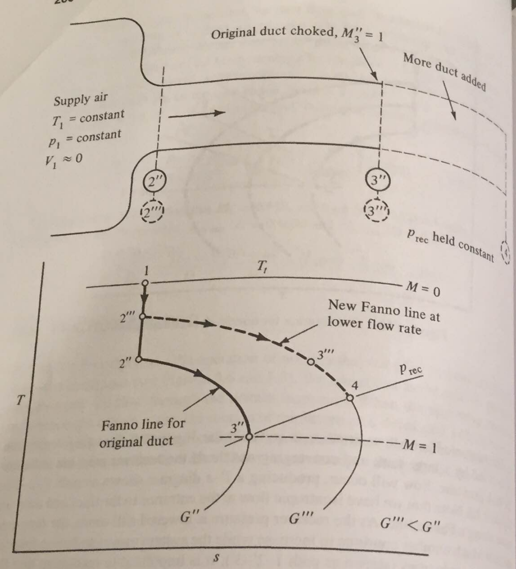

Figure 9.11 Addition of more duct when choked.

the system is no longer choked, although the flow rate has decreased. What would occur if the receiver pressure were now lowered?

In summary, when a *subsonic* Fanno flow has become *friction choked* and more duct is added to the system, the flow rate must decrease. Just how much it decreases and whether or not the exit velocity remains sonic depends on how much duct is added and the receiver pressure imposed on the system.

Now suppose that we are dealing with *supersonic* Fanno flow that is *friction choked*. In this case the addition of more duct causes a normal shock to form inside the duct. The resulting subsonic flow can accommodate the increased duct length at the same flow rate. For example, Figure 9.12 shows a Mach 2.18 flow that has an fL_{max}/D value of 0.356. If a normal shock were to occur at this point, the Mach number after the shock would be about 0.550, which corresponds to an fL_{max}/D

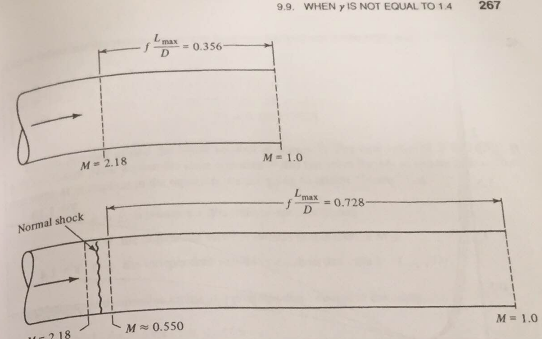

Figure 9.12 Influence of shock on maximum duct length.

value of 0.728. Thus, in this case, the appearance of the shock permits over twice the duct length to the choke point. This difference becomes even greater as higher Mach numbers are reached.

The shock location is determined by the amount of duct added. As more duct is added, the shock moves upstream and occurs at a higher Mach number. Eventually, the shock will move into that portion of the system that precedes the constant-area duct. (Most likely, a converging–diverging nozzle was used to produce the supersonic flow.) If sufficient friction length is added, the entire system will become subsonic and then the flow rate will decrease. Whether or not the exit velocity remains sonic will again depend on the receiver pressure.

9.9. WHEN γ IS NOT EQUAL TO 1.4

As indicated earlier, the Fanno flow table in Appendix I is for $\gamma = 1.4$. The behavior of fL_{max}/D, the friction function, is given in Figure 9.13 for $\gamma = 1.13$, 1.4, and 1.67 for Mach numbers up to $M = 5$. Here we can see that the dependence on γ is rather noticeable for $M \geq 1.4$. Thus, below this Mach number the tabulation in Appendix I may be used with little error for any γ. This means that for subsonic flows, where most Fanno flow problems occur, there is little difference between the various gases. The desired accuracy of results will govern how far you want to carry this approximation into the supersonic region.

Strictly speaking, these curves are only representative for cases where γ variations are *negligible within the flow*. However, they offer hints as to what magnitude of

For a shock to occur as specified, the duct flow must be supersonic, which mea[illegible] the nozzle is operating at its third critical point. The inlet conditions and nozzle are[illegible] fix conditions at location 3. We can then find p^* at the tip of the Fanno line. Then the [illegible] p_5/p^* can be computed and the Mach number after the shock is found from the Fanno [illegible] This solution probably would not have occurred to us had we not drawn the T–s diagram [illegible] recognized that point 5 is on the same Fanno line as 3, 4, and *.

For $A_3/A_2 = 2.4$, $M_3 = 2.4$ and $p_3/p_{t3} = 0.06840$. We proceed immediately to comp[illegible] p_5/p^*:

$$\frac{p_5}{p^*} = \frac{p_5}{p_{t1}}\frac{p_{t1}}{p_{t3}}\frac{p_{t3}}{p_3}\frac{p_3}{p^*} = \left(\frac{3}{8}\right)(1)\left(\frac{1}{0.0684}\right)(0.3111) = 1.7056$$

From the Fanno table we find that $M_5 = 0.619$, and then from the shock table, $M_4 = 1$[illegible] Returning to the Fanno table, $fL_{3\,\max}/D = 0.4099$ and $fL_{4\,\max}/D = 0.2382$. Thus

$$\frac{f\,\Delta x}{D} = \frac{fL_{3\,\max}}{D} - \frac{fL_{4\,\max}}{D} = 0.4099 - 0.2382 = 0.172$$

9.8 FRICTION CHOKING

In Chapter 5 we discussed the operation of nozzles that were fed by constant stagna[illegible] tion inlet conditions (see Figures 5.6 and 5.8). We found that as the receiver pressure was lowered, the flow through the nozzle increased. When the *operating pressure ratio* reached a certain value, the section of minimum area developed a Mach number of unity. The nozzle was then said to be choked. Further reduction in the pressure ratio did not increase the flow rate. This was an example of *area choking*.

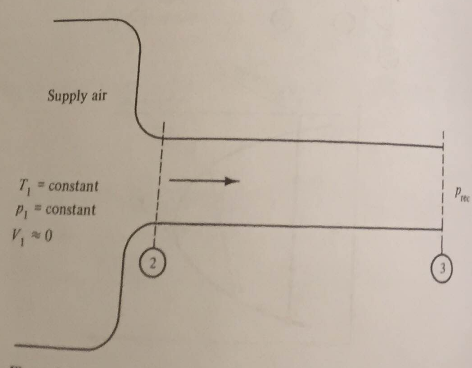

Figure 9.9 Converging nozzle and constant-area duct combination.

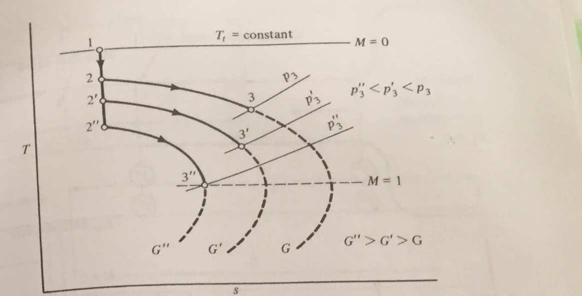

Figure 9.10 T–s diagram for nozzle–duct combination.

The subsonic Fanno flow situation is quite similar. Figure 9.9 shows a given length of duct fed by a large tank and converging nozzle. If the receiver pressure is below the tank pressure, flow will occur, producing a T–s diagram shown as path 1–2–3 in Figure 9.10. Note that we have isentropic flow at the entrance to the duct and then we move along a Fanno line. As the receiver pressure is lowered still more, the flow rate and exit Mach number continue to increase while the system moves to Fanno lines of higher mass velocities (shown as path 1–2′–3′). It is important to recognize that the receiver pressure (or more properly, the operating pressure ratio) is controlling the flow. This is because in subsonic flow the pressure at the duct exit must equal that of the receiver.

Eventually, when a certain pressure ratio is reached, the Mach number at the duct exit will be unity (shown as path 1–2″–3″). This is called *friction choking* and any further reduction in receiver pressure would not affect the flow conditions *inside* the system. What would occur as the flow leaves the duct and enters a region of reduced pressure?

Let us consider this last case of choked flow with the exit pressure equal to the receiver pressure. Now *suppose that the receiver pressure is maintained at this value* but more duct is added to the system. (Nothing can physically prevent us from doing this.) What happens? We know that we cannot move *around the Fanno line*, yet somehow we must reflect the added friction losses. This is done by moving to a new Fanno line at a *decreased* flow rate. The T–s diagram for this is shown as path 1–2‴–3‴–4 in Figure 9.11. Note that pressure equilibrium is still maintained at the exit but

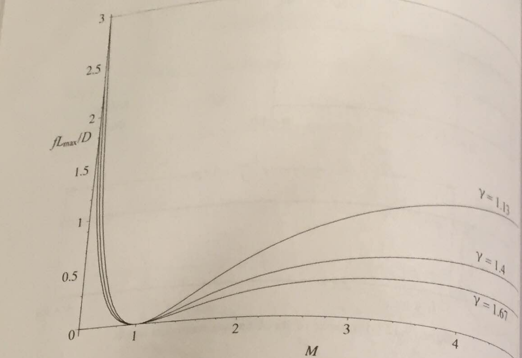

Figure 9.13 Fanno flow fL_{max}/D versus Mach number for various values of γ.

changes are to be expected in other cases. Flows where γ variations are *not negligible within the flow* are treated in Chapter 11.

9.10 (OPTIONAL) BEYOND THE TABLES

As pointed out in Chapter 5, one can eliminate a lot of interpolation and get accurate answers for any ratio of the specific heats γ and/or any Mach number by using a computer utility such as MAPLE. This utility is useful in the evaluation of equation (9.46). Example 9.6 is one such application.

Example 9.6 Let us rework Example 9.3 without using the Fanno table. For $M_1 = 0.36$, calculate the value of fL_{max}/D. The procedure follows equation (9.46):

$$\frac{f(x^* - x)}{D_e} = \left(\frac{\gamma+1}{2\gamma}\right) \ln \frac{[(\gamma+1)/2]M^2}{1+[(\gamma-1)/2]M^2} + \frac{1}{\gamma}\left(\frac{1}{M^2} - 1\right) \qquad (9.46)$$

Let

$g \equiv \gamma$, a parameter (the ratio of specific heats)

$X \equiv$ the independent variable (which in this case is M_1)

$Y \equiv$ the dependent variable (which in this case is fL_{max}/D)

Listed below are the precise inputs and program that you use in the computer.

```
[> g := 1.4:    X := 0.36:
[> Y := ((g + 1)/(2*g))*log(((g + 1)*(X^2)/2)/(1 +
   (g - 1)*(X^2)) + (1/g)*((1/X^2) - 1);
                    Y: = 3.180117523
```

We can proceed to find the Mach number at station 2. The new value of Y is 3.1801 − 2.772 = 0.408. Now we use the same equation (9.46) but solve for M_2 as shown below. Note that since M is implicit in the equation, we are going to utilize "fsolve". Let

$g \equiv \gamma$, a parameter (the ratio of specific heats)

$X \equiv$ the dependent variable (which in this case is M_2)

$Y \equiv$ the independent variable (which in this case is fL_{max}/D)

Listed below are the precise inputs and program that you use in the computer.

```
[> g2 := 1.4:    Y2 := 0.408:
[> fsolve(Y2 = ((g2 + 1)/(2*g2))*log(((g2 + 1)*(X2^2)/2)/(1 +
   (g2 - 1)*(X2^2)/2)) + (1/g2)*((1/X2^2) - 1), X2, 0..1);
                    .6227097475
```

The answer of $M_2 = 0.6227$ is consistent with that obtained in Example 9.3. We can now proceed to calculate the required static properties, but this will be left as an exercise for the reader.

9.11 SUMMARY

We have analyzed flow in a constant-area duct with friction but without heat transfer. The fluid properties change in a predictable manner dependent on the flow regime as shown in Table 9.3. The property variations in subsonic Fanno flow follow an intuitive pattern but we note that the supersonic flow behavior is completely different. The

Table 9.3 Fluid Property Variation for Fanno Flow

Property	Subsonic	Supersonic
Velocity	Increases	Decreases
Mach number	Increases	Decreases
Enthalpy[a]	Decreases	Increases
Stagnation enthalpy[a]	Constant	Constant
Pressure	Decreases	Increases
Density	Decreases	Increases
Stagnation pressure	Decreases	Decreases

[a] Also temperature if the fluid is a perfect gas.