Hamilton's principle and Fermat's principle are only examples showing that the physical universe follows paths through space and time based on extrema principles. Almost in all branches of physics, one can find such a principle. Why the nature operates in accordance with this principle of economy is a question for philosophers and theologians. As scientists, we can just enjoy the elegance of the theory. However this is not to say that variation principle is merely a device to provide an alternative derivation of known results. In fact its impact on the development of science cannot be overemphasized. When the basic physics is not yet known, a postulated variational principle can be very useful. A shining example is Richard Feynman's formulation of quantum electrodynamics which is based on the principle of least action. For his achievements, he was awarded a 1965 Nobel prize in physics.

Variational principle as a computation tool is also very important. With variational methods, energy levels of all kinds of molecules can now be calculated to a high degree of accuracy. John Pople codified such calculations in a computer program known as GAUSSIAN. He was awarded a 1998 Nobel prize in chemistry.

Exercises

1. Find the Euler–Lagrange equation for
$$\text{(a) } F = x^2y^2 - y'^2, \quad \text{(b) } F = \sqrt{xy} + y'^2.$$
Ans. (a) $y'' + x^2y = 0$, (b) $\dfrac{1}{4}\sqrt{\dfrac{x}{y}} - y'' = 0$.

2. Find the curve $y(x)$ that will make the following functional stationary
$$\text{(a) } I = \int_a^b (y^2 + y'^2 + 2y\mathrm{e}^x)\mathrm{d}x,$$
$$\text{(b) } I = \int_a^b \frac{y'^2}{x^3}\mathrm{d}x.$$
Ans. (a) $y = \frac{1}{2}x\mathrm{e}^x + c_1\mathrm{e}^x + c_2\mathrm{e}^{-x}$, (b) $y = c_1x^4 + c_2$.

3. Find the function $y(x)$ that passes through the points $(0,0)$ and $(1,1)$ and minimizes
$$I(y) = \int_0^1 (y^2 + y'^2)\mathrm{d}x.$$
Ans. $y(x) = 0.42\mathrm{e}^x - 0.42\mathrm{e}^{-x}$.

4. Find the function $y(x)$ that passes through the points $(0,0)$ and $(\pi/2,1)$ and minimizes

$$I(y) = \int_0^1 (y'^2 - y^2)\mathrm{d}x.$$

Ans. $y(x) = \sin x$.

5. What would be the functional corresponding to the following problem:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 1, \quad 0 < x < 1,\ 0 < y < 1,$$
$$u = 0, \quad \text{on the boundary.}$$

Ans. $I(u) = \int_0^1 \int_0^1 \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + 2u \right] \mathrm{d}x\mathrm{d}y$.

6. Show that if the integrand of the following integral:

$$I = \int_{t_1}^{t_2} F(x, y, x', y')\mathrm{d}t$$

does not explicitly contain the independent variable t, then the Euler–Lagrange equations lead to

$$F - x'\frac{\partial F}{\partial x'} - y'\frac{\partial F}{\partial y'} = C,$$

where C is a constant.

7. Find the Euler–Lagrange equation for the functional

$$I = \int_0^1 (yy'' + 4y)\mathrm{d}x.$$

Ans. $y'' + 2 = 0$.

8. Find the Euler–Lagrange equation for the functional

$$I = \int_0^1 (-y'^2 + 4y)\mathrm{d}x.$$

Ans. $y'' + 2 = 0$.

9. Show that the Euler–Lagrange equation for the three-dimensional functional

$$I = \iiint \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 \right] \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

is given by the Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

10. Estimate the lowest vibrational frequency of a circular drum-head with radius a, using the functional

$$\frac{\omega^2}{v^2} = \frac{-\int\int u\nabla^2 u\,\mathrm{d}x\mathrm{d}y}{\int\int u^2\mathrm{d}x\mathrm{d}y}$$

and the trial function

$$u(r) = r - a.$$

Ans. $\omega = 2.449v/a$.

11. If $I[u]$ and $J[u]$ are both two-dimensional functionals and

$$\lambda[u] = \frac{I[u]}{J[u]},$$

show that to minimize $\lambda[u]$ is equivalent to minimizing the functional $K[u]$

$$K[u] = I[u] - \lambda J[u].$$

Hint: Replace $u(x,y)$ by $U(x,y) + \alpha\eta(x,y)$, and show that $\left.\frac{\mathrm{d}\lambda}{\mathrm{d}\alpha}\right|_{\alpha=0} = 0$ leads to $\left[\frac{\mathrm{d}I}{\mathrm{d}\alpha} - \lambda\frac{\mathrm{d}J}{\mathrm{d}\alpha}\right]_{\alpha=0} = 0.$

12. Find the Euler–Lagrange equation for the functional

$$I = \int_0^1 xy'^2\mathrm{d}x$$

subject to the constraint

$$\int_0^1 xy^2\mathrm{d}x = 1.$$

Ans. $xy'' + y' - \lambda xy = 0$.

13. Find the Euler–Lagrange equation for the functional

$$I = \int_0^1 (py'^2 - qy^2)\mathrm{d}x$$

subject to the constraint

$$\int_0^1 ry^2\mathrm{d}x = 1.$$

Ans. $\frac{\mathrm{d}}{\mathrm{d}x}(py') + (q - \lambda r)y = 0$.